WEEK 6:

→ determines the topology on the localization.

A non-arch. $T_i \subseteq A$ $(i \in I)$

Condition: (*) $\forall A \supseteq_{open} U \ni 0$, $\forall i \in I$, $\forall n \in \mathbb{N}$, $T_i^n . U =$ "additive subgroup gen'd by $\{\prod a_j . u \in A \mid a_j \in T_i, u \in A\}$" is open

Suppose $(f_1 \ldots f_n)$ is open $R\left(\frac{f_1 \ldots f_n}{g}\right) = \{x \in Spa(A, A^+) \mid \forall i \ |f_i|_x \leq |g_i|_x \neq 0\}$.
In fact $(f_1 \ldots f_n) = \underbrace{\{f_1 \ldots f_n\}}_{T_1} . A$ as above.

Main example: $R\left(\frac{X, p}{p}\right) \subseteq Spa(\mathbb{Q}_p\langle X\rangle, \mathbb{Z}_p\langle X\rangle)$

First: non-arch localization $A\left(\frac{f_1 \ldots f_n}{g}\right)$ ← T_1 ← S_1

Proposition: If also $S_i \in A$ $R := \{$mult. set gen. by the $s_i\}$. Then $\exists!$ non-arch. ring structure on $R^{-1}A$ st for $\varphi : A \to R^{-1}A$ structure map,

(a) φ continuous & $\{\varphi(t)/\varphi(s_i) \in R^{-1}A \mid t \in T_i\}$ is power bounded. (the topology of $R^{-1}A$ is such that $\langle \varphi(t)/\varphi(s_i)\rangle_{subring}$ is bounded)

Notation: $A(T_i/s_i \mid i \in I) = R^{-1}A$

(b)

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\forall f \text{ contin.}"'] & A(T_i/s_i \mid i \in I) \arrow[d, dashed, "\exists! \text{ continuous}"] \\ & B \text{ non arch.} \end{tikzcd}$$

such that $\left\{\frac{f(t_i)}{f(s_i)} \mid t \in T_i, i \in I\right\}$ power bounded & $f(s_i) \in B^\times$.

Remark: for $R\left(\frac{X, p}{p}\right)$ $T_1 = \{x_1, p\}$ $S_1 = \{p\}$
$A = \mathbb{Q}_p\langle x\rangle$ $R^{-1}A = \mathbb{Q}_p\langle x\rangle$ but different topology.

Remark: Recall T bounded if $\forall$ open $U \ni 0$ $\exists$ open $V \ni 0$ st $T.V \subseteq U$
A non-arch. $\Rightarrow$ WMA U subgroup $\Rightarrow$ (T bounded $\Leftrightarrow$ $\langle T\rangle_+$ bounded).

Proposition: A non-arch. $T \subseteq A$. Then T power bounded $\Leftrightarrow$ $\langle T\rangle_{subring}$ bounded.

In our example, $\left\langle \frac{\varphi(t)}{\varphi(s_i)} \mid t \in T_1 \right\rangle_{subring} = \underbrace{\mathbb{Z}\left[\frac{X}{p}\right]}_{\text{"}D\text{"}} \subseteq \underbrace{\mathbb{Q}\langle X\rangle}_{\text{"}A\text{"}}$. Notice that $p^n . \mathbb{Z}_p\langle X\rangle . D \not\subseteq D$

So: The opens of $R^{-1}A$ will be of the form $D.U$ where U is an open of A.

Lemma: A ring $\mathcal{G} \subseteq 2^A$ a set of additive subgroups. Then $\mathcal{G}$ is a set of fundamental neighbourhoods of 0 if and only if

($\{a + G \mid a \in A \ G \in \mathcal{G}\}$ basis of top (**))

(i) $\forall G, G' \in \mathcal{G} \Rightarrow \exists H \in \mathcal{G}$ st $H \subseteq G \cap G'$.
(ii) $\forall a \in A$, $\forall G \in \mathcal{G}$ $\exists H \in \mathcal{G}$: $Ha \subseteq G$
(iii) $\forall G \in \mathcal{G}$ $\exists H \in \mathcal{G}$: $H.H \subseteq G$.

Proof: (**) $\Rightarrow$ (i) ✓, (ii) ✓ as $\cdot a$ is continuous, (iii) mult. is continuous.
(i) $\Rightarrow$ $\mathcal{G}$ generates a topology. We need (i)+(ii) $\Rightarrow$ mult. & addition is cont.
mult. continuous: $b + G$ open want to show $\{(x, y) \in A \times A \mid x.y + b \in G\}$ is open.
Fix (x, y) st $x.y \in G + b$ then by (iii) $\exists H' \in \mathcal{G}$ st $H'.H' \subseteq G$
(ii) $\exists H'', H''' \in \mathcal{G}$ st $H''.x \subseteq G$, $H'''.y \subseteq G$.
Take $H := H' \cap H'' \cap H'''$ still open. Then $(x + H).(y + H) \subseteq G \Rightarrow (x+H) \times (y+H)$ is a neigh. of (x, y).

Proof of the localization proposition: $D = \langle t/s_i \mid t \in T_i, i \in I\rangle_{subring} \subseteq A$.
$\mathcal{G} = \{D.U \mid U$ additive open neighb. of $0 \subseteq A\}$. We verify that the properties (i), (ii) (iii) of the above lemma holds for $\mathcal{G}$.
(i): $G = D.U$, $G' = D.U'$ as A non-arch. $\exists V \subseteq U \cap U'$ such that $D.V \subseteq D.U \cap D.U'$.
(iii): is similar. (ii): Fix $a/s \in R^{-1}A$, $s = \prod s_i^{n_i}$ and fix $D.U \in \mathcal{G} \Rightarrow \exists V : V.a \subseteq U$
$\Rightarrow a.D.V \subseteq D.U$. We know that $\exists T/s := \prod T_i^{n_i}/s_i^{n_i} \subseteq D$ where $t_i \in T$
Note that $a.D.V \supseteq a.\frac{T}{s}.D.V$ as $T/s \in D$ and by (*) $T.V$ is open thus
$\frac{a}{s}.D.(T.V) \subseteq D.U$.

In our example, $D = \mathbb{Z}[\frac{x}{p}] \subseteq \mathbb{Q}_p\langle x\rangle$, fund. system of neigh. of 0: $\mathbb{Z}[\frac{x}{p}].p^n.\mathbb{Z}_p\langle x\rangle$
$= p^n.\mathbb{Z}_p\langle x\rangle[\frac{x}{p}]$.

Continuation of the proof: φ is continuous: need $\varphi^{-1}(D.\varphi(U)) \supseteq U$ ✓
D is bounded by the choice of the topology $\Rightarrow$ powerboundedness ✓ $\Rightarrow$ (a) ✓
For (b) g exists uniquely as a ring homo we need g continuous.
Let $E = \langle f(t)/f(s_i) \mid t \in T_i \rangle_{\text{subring}} \subseteq B$ is bounded & $g(D) = E$. Fix $U \subseteq B$ open subgroup
$\xRightarrow[E \text{ bdd}]{} \exists V$ open subgroup $\subseteq B$ st $V.E \subseteq U \xRightarrow[f \text{ cont.}]{} \exists W$ open subgroup $\subseteq A: W \subseteq f^{-1}(V)$
$\Rightarrow g(W.D) = g(W).g(D) \subseteq V.E \subseteq U$ ∎

Back to the example: Is $\sum_{i=0}^{\infty} x^i$ supposed to be conv. if A was complete i.e is $\{x^i\}$ topologically nilpotent? $x^i \in p^i \mathbb{Z}_p\langle x\rangle[\frac{x}{p}] \ni p^i (\frac{x}{p})^i = x^i$ but
$\Rightarrow$ top. nilpotent in localization! $\quad \sum_{i=0}^{\infty} x^i \notin \mathbb{Q}_p\langle x\rangle \cong_{\text{rings}} \mathbb{Q}_p\langle x\rangle(\frac{p.x}{p})$
$\Rightarrow \mathbb{Q}_p\langle x\rangle(\frac{p.x}{p})$ is not complete!

Two Questions:
(1) Does this non-arch. localization give f-adic ring if we start w/ f-adic ring?
(2) Completion of f-adic rings.

Definition: Assume A f-adic $\Rightarrow A\langle \frac{T_1}{S_1} \rangle =$ completion of $A(\frac{T_1}{S_1})$

Completion for $\mathbb{Z}_p\langle x\rangle[\frac{x}{p}]$ with p-adic topology is $\varprojlim \mathbb{Z}_p\langle x\rangle[\frac{x}{p}]/p^n$ so we should get
$\mathbb{Q}_p\langle x\rangle\langle \frac{p.x}{p}\rangle \cong \left(\varprojlim \mathbb{Z}_p\langle x\rangle[\frac{x}{p}]/p^n\right)[\frac{1}{p}]$.